Reminder: - Midterm Wed 2/27 in class.
(1:00 - 1:50)

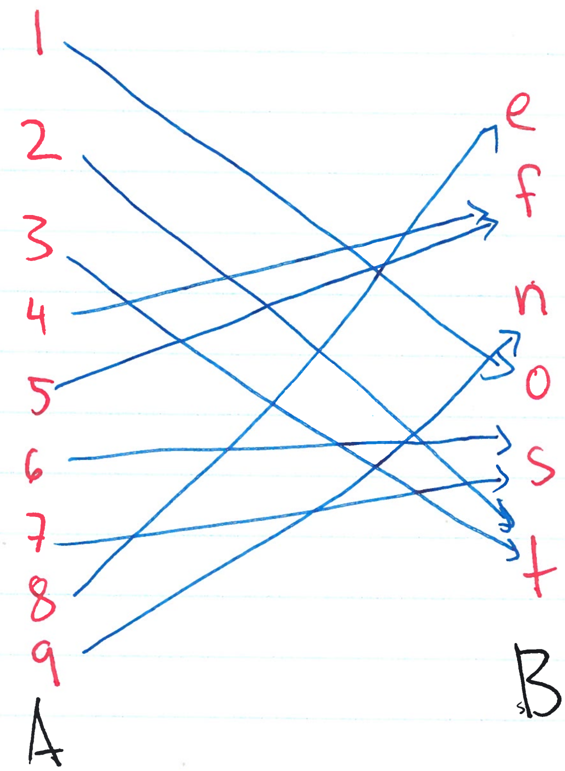
- Covers the same material as before.

Functions: Example: Consider the function which takes a number from 1 to 9

$\{1,2,3,4,5,6,7,8,9\} \longrightarrow \{e,f,n,o,s,t\}$

and gives its first letter. } One way to describe the function

Another description of the function



A third way to describe the function is as the set

$\{(1,o),(2,t),(3,t),(4,f),(5,f),(6,s),(7,s),(8,e),(9,n)\}.$

Def: Let A and B be sets. A function from A to B is a subset of the Cartesian product

$f \subseteq A \times B$ with two properties

Every input gives an output
- For every $a \in A$, there is some $b \in B$ such that $(a,b) \in f$

That output is unique.
- If (a,b) and (a,b') are both elements of f, then $b = b'$.

If $(a,b) \in f$, then we write $f(a) = b$.

(The set $f \subseteq A \times B$ is actually the "graph" of the function.)

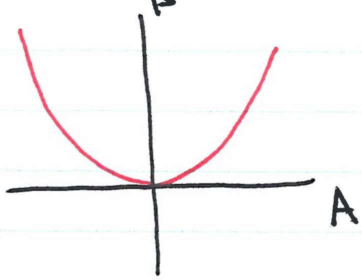
The set A is called the domain
——— B ——— codomain

The range is the set $\{b \in B : \exists a \in A, f(a) = b\}$.

(In general, $\text{range}(f) \subseteq B$)

Example: $A = \mathbb{R}$, $B = \mathbb{R}$

Consider $f: A \to B$ defined by $f(x) = x^2$



Example: $A = \{1,2,3,4,5,6\}$, $B = \{1,2,3,4,5,6\}$

$f: A \to B$ defined by $f(a) =$ "take 3^a and reduce mod 7 until in $\{1,2,3,\dots,6\}$."

$1 \longmapsto 3^1 = 3$

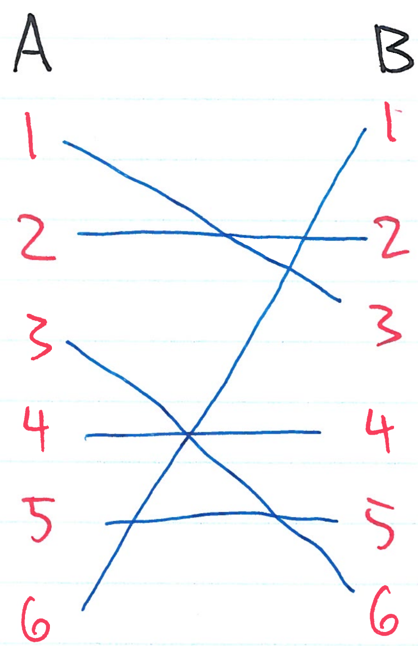
$2 \longmapsto 3^2 = 9 \equiv 2 \bmod 7$

$3 \longmapsto 3^3 = 27 \equiv 6 \bmod 7$

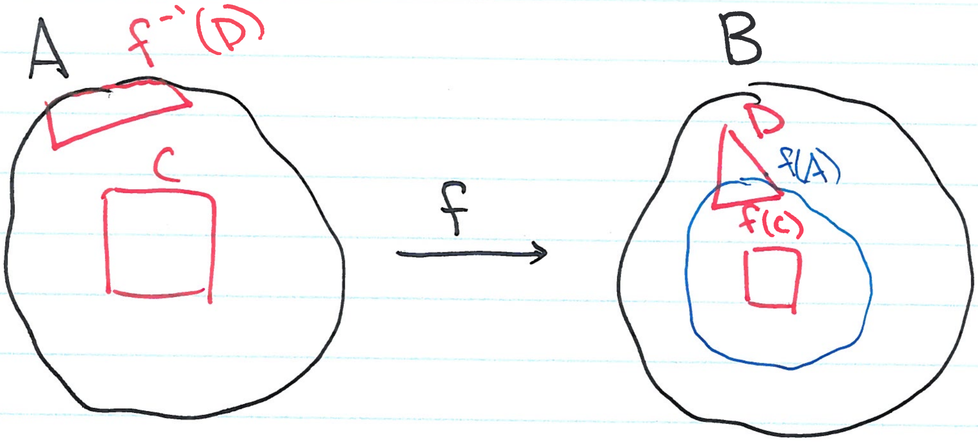
$4 \longmapsto 3^4 = 81 \equiv 4 \bmod 7$

$5 \longmapsto 243 \equiv 5 \bmod 7$

$6 \longmapsto 729 \equiv 1 \bmod 7$



Let $f: A \longrightarrow B$ (f is a function from A to B.)



<u>Def</u>: Let C be a subset of A. The <u>image</u> of C is the set $\{b \in B : \exists a \in C,\ f(a) = b\}$

$(f(C))$ "the elements hit by C.

<u>Def</u>: Let D be a subset of B. The <u>preimage</u> of D is the set $\{a \in A : f(a) \in D\}$.

$f^{-1}(D)$

<u>Caution</u>: "f^{-1}" is not exactly the inverse function to f. In fact, you don't always have an inverse!

<u>Exercise:</u> Let $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$.

Compute the following sets

- $f([0,4]) = [0,16]$
- $f^{-1}([0,9]) = [-3,3]$
- $f([-1,2]) = [0,4]$
- $f^{-1}([1,4]) = [-2,-1] \cup [1,2]$

<u>Easy to prove:</u>

- If C_1 and C_2 are subsets of A and $C_1 \subseteq C_2$, then $f(C_1) \subseteq f(C_2)$
- ~ D_1 ~ D_2 ——— B and $D_1 \subseteq D_2$, then $f^{-1}(D_1) \subseteq f^{-1}(D_2)$
- For every subset $C \subseteq A$, $f^{-1}(f(C)) \supseteq C$

not necessarily equal!

- ——— $D \subseteq B$, $f(f^{-1}(D)) \subseteq D$

Prop: For every $C \subseteq A$, $f^{-1}(f(C)) \supseteq C$

Pf: We must show that $\forall x$, $x \in C \Rightarrow x \in f^{-1}(f(C))$.

If $x \in C$, then $f(x) \in f(C)$, and thus by definition, $x \in f^{-1}(f(C))$. ∎

Note: It is not true that $f^{-1}(f(C)) = C$. For example, consider $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Let $C = [0,4]$

Then $f^{-1}(f([0,4])) = f^{-1}([0,16]) = [-4,4]$.

Prop: For every $D \subseteq B$, $f(f^{-1}(D)) \subseteq D$

Pf: We must show that $\forall x$, $x \in f(f^{-1}(D)) \Rightarrow x \in D$

If $x \in f(f^{-1}(D))$, then there is some $y \in f^{-1}(D)$ such that $f(y) = x$.

But since $y \in f^{-1}(D)$, we also have $f(y) \in D$.

Thus, $x \in D$. ∎

Note: Consider $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Let $D =$ any set of negative numbers.

Then $f(f^{-1}(D)) = f(\emptyset) = \emptyset$

In general, if D contains negative numbers then $f(f^{-1}(D))$ cannot equal D.